Finite-size scaling of corner transfer matrices for the two-dimensional Ising model

1988 J. Phys. A: Math. Gen. 21 L185

(http://iopscience.iop.org/0305-4470/21/3/012)

J. Phys. A: Math. Gen. **21** (1988) L185-188. Printed in the UK

LETTER TO THE EDITOR

Finite-size scaling of corner transfer matrices for the two-dimensional Ising model

Ingo Peschel

Fachbereich Physik, Freie Universität Berlin, Arnimallee 14, D-1000 Berlin 33, Federal Republic of Germany

Received 9 November 1987

Abstract. The eigenvalue spectra of generalised corner transfer matrices are calculated for an Ising model on a square lattice. The predictions of conformal invariance at the critical point are thereby verified.

Corner transfer matrices (CTM) are efficient tools for calculating the spontaneous order in exactly solvable models [1]. In these cases their eigenvalue spectrum was found to be remarkably simple. Writing the eigenvalues as $\exp(-E_n)$, the lowest E_n are equidistant at all temperatures $T < T_c$ with the spacing vanishing at the critical point. This was found analytically for the limiting case of an infinite lattice.

At the critical point, on the other hand, the finite-size properties of a system are usually more interesting. Here conformal symmetry makes various predictions about the properties of conventional row-to-row transfer matrices [2]. It has been shown recently that the same is true for CTM [3]. In this case one considers a system of the shape shown in figure 1(a). The order parameter is fixed (or free) along the inner and outer circular boundaries and the transfer matrix runs in the azimuthal direction. For the Ising model the E_n are sums of single-particle energies ε_l and the prediction of conformal invariance is that the lowest ε_l are of the form $\varepsilon_l = (2l-1)\varepsilon$ where $l \geqslant 1$ and

$$\varepsilon = \Theta \frac{\pi/2}{\ln(R/r)}. \tag{1}$$

The peculiar size dependence of ε distinguishes it from the corresponding result for the row-to-row transfer matrix. It has been checked to some extent in [3] for the case

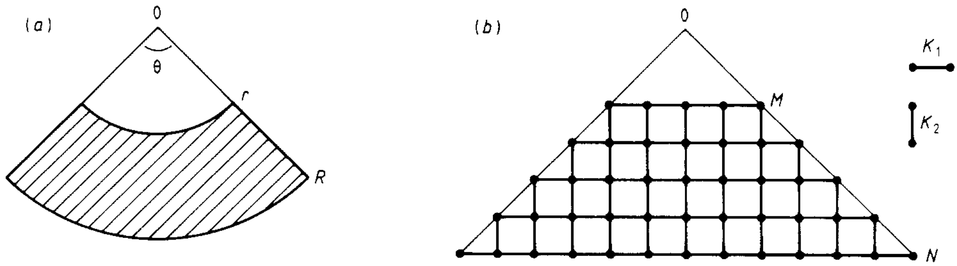

Figure 1. Shape of the system: (a) in the continuum limit; (b) for the case of a square lattice with a 90° corner.

of a full corner, i.e. $r \to 0$. In this letter, general values of the inner radius r will be considered. This corresponds to corners with some inner portion taken out. CTM for such a situation have not been studied before. They are in a sense intermediates between Baxter's original CTM and the usual row-to-row transfer matrices. It turns out that they are much better suited to verify (1) than the usual CTM.

The lattice geometry for the numerical calculation is shown in figure 1(b). It is a square lattice with a 90° corner and radial boundaries along the lattice diagonals. There is a slight difference to figure 1(a), because the inner and outer boundaries are not curved. Calculations were done for the Hamiltonian limit (weak vertical couplings K_2 and strong horizontal couplings K_1) and also for the isotropic case. The first case is much simpler. Writing the CTM as $A = \exp(-K_1^* H)$, the operator H can be read off to be

$$H = -\sum_{n=M+1}^{N-1} 2n\sigma_n^z - \lambda \sum_{n=M}^{N-1} (2n+1)\sigma_n^x \sigma_{n+1}^x. \tag{2}$$

Here the σ_n^α are Pauli matrices, K_1^* is the dual coupling of K_1 ($\tanh K_1^* = \exp(-2K_1)$) and $\lambda = K_2/K_1^*$. The critical point corresponds to $\lambda = 1$. After introducing fermions, H can be diagonalised numerically by standard techniques [4], giving

$$H = \sum_l \omega_l \alpha_l^+ \alpha_l + \text{constant} \tag{3}$$

where α_l, α_l^+ are Fermi operators. Due to the fixed boundary conditions one ω_l is zero, leaving $N - M$ non-trivial single-particle eigenvalues $\omega_l = \varepsilon_l / K_1^*$.

For a full corner ($M = 0$) the spectrum of H has already been determined [3]. Plotting ω_l against $(2l-1)$ the points follow a linear relation for small l if one is below the critical temperature. At the critical point, however, such a linear regime is barely visible even for $N \sim 100$. The comparison with the conformal result is therefore somewhat difficult. This changes for $M > 0$. Then the M smallest ω-values of the full corner problem disappear and the smallest remaining ω are shifted downwards. This destroys the regular spacing of the lowest eigenvalues if $T < T_c$, but it creates a linear regime right at T_c. This is seen in figure 2 where spectra at the critical point for fixed

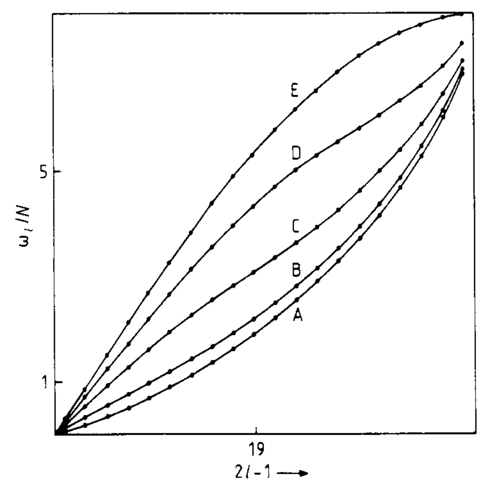

Figure 2. Single-particle spectrum of the operator H, equation (2), at the critical point for $N - M = 20$, and various ratios of N/M: A, 20/0; B, 22/2; C, 30/10; D, 40/20. Spectrum E is for $N, M \to \infty$. The curves are guides for the eye.

$N-M=20$ and various ratios of N/M are shown. Curve A corresponds to the full corner and curve E to the limit $N/M \to 1$. In this case one is effectively dealing with a strip of width $(N-M)$ and length $(N+M)$. H then becomes $(N+M)$ times the Hamiltonian of the Ising chain in a transverse field, the spectrum of which is well known [5].

If one increases the size of the system at fixed ratio N/M more eigenvalues fall into the linear regions and the lowest ones follow the linear law better and better. This is illustrated in table 1 for the case $N/M=5$. One also sees that the value of $\omega=\omega_1$ slowly increases with N. Closer inspection shows that it follows the law $\omega(N)=\omega(\infty)-b/N+\mathrm{O}(1/N^2)$ where b depends on N/M and $\omega(\infty)$ coincides numerically with the conformal result

$$\omega=\frac{2\pi}{\ln(N/M)}. \tag{4}$$

Equation (4) is obtained by rescaling the anisotropic system so that it becomes effectively isotropic [6]. For this one uses the ratio of the correlation lengths along the two axes which near T_c is $\xi_2/\xi_1=\cosh 2K_2/\cosh 2K_1$. The angle $\Theta=\pi/2$ then becomes an effective angle $\Theta_{\mathrm{eff}}=4K_1^* \ll 1$. Using this in (1) and $r=M$, $R=N$ gives the result for ω. Actually, one should take $r<M$ and $R>N$ since in the continuum theory the order parameter diverges at the boundaries. This would give a $1/N$ correction to (4). Similarly, the rescaling will contain modifications in a small system. To avoid these complications, one should only compare the asymptotic values of ω for the lattice and the continuum. From their coincidence one concludes that the shape of the boundaries is not important here. This is plausible in view of the small effective angle.

Table 1. Lowest single-particle eigenvalues of H, equation (2), for $\lambda=1$ and $N/M=5$. Shown are $\omega=\omega_1$ and the ratios $\nu_n=\omega_n/\omega$.

M N	3 15	6 30	9 45	12 60	Conformal result (4)
ω	3.6790	3.7870	3.8250	3.8444	3.9040
ν_1	1.0000	1.0000	1.0000	1.0000	1.0000
ν_2	2.9718	2.9923	2.9965	2.9980	3.0000
ν_3	4.8604	4.9613	4.9826	4.9902	5.0000
ν_4	6.6318	6.8904	6.9510	6.9724	7.0000
ν_5	8.3337	8.7618	8.8942	8.9407	9.000

The calculations for an isotropic system are more difficult. In this case the CTM is diagonalised using the techniques of Kaufman [7] and Abraham [8]. This leads to a $2(N-M)\times 2(N-M)$ matrix R which has the eigenvalues $\exp(\pm\varepsilon_l)$. It is obtained as a product of corresponding matrices R_n which are associated with the row-to-row transfer matrices from which the CTM can be built. The form of these R_n is given in [8]. It is no problem to set up R, but the wide range of the eigenvalues causes numerical problems. Therefore only small systems have been studied, using a Lanczos method. Some results are presented in table 2 together with those of the Hamiltonian limit. One notices that in those examples the isotropic systems show a better linear spacing and the absolute values are closer to the conformal result. That this should be so is not obvious. One might think that the straight boundaries of the lattice system would

Table 2. Lowest single-particle eigenvalues for isotropic systems (ε) and anisotropic systems (ω), together with the conformal results and the ratios $\nu_n = \varepsilon_n/\varepsilon_1 = \omega_n/\omega_1$.

	$M=2, N=8$		$M=2, N=10$	
	Isotropic	H-Limit	Isotropic	H-Limit
ε, ω	1.7016	4.0826	1.4823	3.5808
Conf	1.7799	4.5324	1.5331	3.9040
ν_1	1.0000	1.0000	1.0000	1.0000
ν_2	3.0006	2.9253	3.0026	2.9472
ν_3	5.0200	4.7099	5.0187	4.7741
ν_4	7.1867	6.5563	7.1096	6.5717
ν_5	9.7703	8.7852	9.4259	8.5581

matter more in the isotropic case where the opening angle Θ is large, but this does not seem to be the case.

To summarise, the essential features of the eigenvalue spectrum of generalised CTM have been presented and the predictions of conformal invariance have been verified with great accuracy. One might have expected that cut-off corners are better suited for such comparison than full ones, since their inner part can be better represented by a continuum model. All considerations were confined to the Ising model but other systems like the Potts model can in principle be investigated along the same lines.

I would like to thank J Cardy, M Karowski and T T Truong for discussions, G Stollhoff and W v d Linden for help with the numerics and the Max-Planck-Institut für Festkörperforschung, Stuttgart, for its hospitality.

References

[1] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
[2] Cardy J L 1986 *Nucl. Phys.* B **270** [FS 16] 186; 1986 *Nucl. Phys.* B **275** [FS 17] 200
[3] Peschel I and Truong T T 1987 *Z. Phys.* B **69** 385
[4] Lieb E H, Schulz T D and Mattis D C 1961 *Ann. Phys., NY* **16** 406
[5] Pfeuty P 1970 *Ann. Phys., NY* **57** 79
[6] Barber M N, Peschel I and Pearce P A 1984 *J. Stat. Phys.* **37** 497
[7] Kaufman B 1949 *Phys. Rev.* **76** 1232
[8] Abraham D B 1971 *Stud. Appl. Math.* **50** 71